Convex Optimization

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2. Convex sets

why convex sets first? it's the space we're optimizing in.
you'll see in 2 slides.

Outline

Some standard convex sets

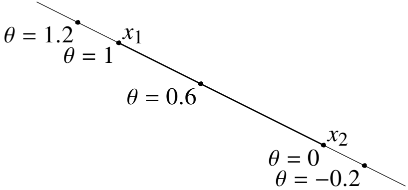
Operations that preserve convexity

Generalized inequalities

Separating and supporting hyperplanes

Affine set

line through x_1, x_2: all points of form $x = \theta x_1 + (1 - \theta)x_2$, with $\theta \in \mathbf{R}$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$
(conversely, every affine set can be expressed as solution set of system of linear equations)

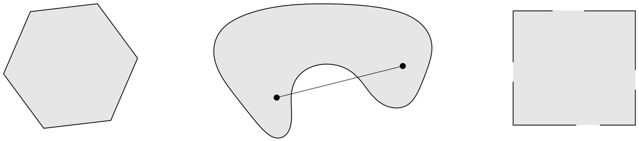
Convex set

line segment between x_1 and x_2: all points of form $x = \theta x_1 + (1-\theta)x_2$, with $0 \le \theta \le 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



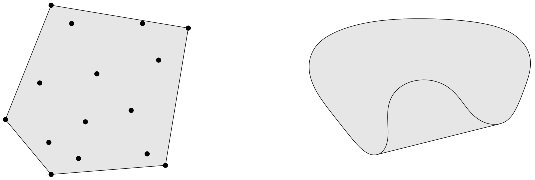
Convex combination and convex hull

convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\textbf{conv}\, S$: set of all convex combinations of points in S

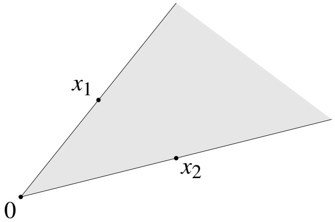


Convex cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

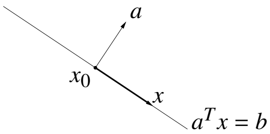
with $\theta_1 \geq 0$, $\theta_2 \geq 0$



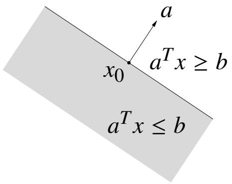
convex cone: set that contains all conic combinations of points in the set

Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$, with $a \neq 0$



halfspace: set of the form $\{x \mid a^T x \leq b\}$, with $a \neq 0$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex check -- why?

Euclidean balls and ellipsoids

(Euclidean) ball with center x_c and radius r:

notation for 2-norm on vectors.

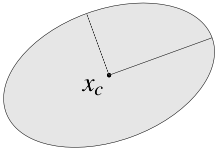
$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)

what is P for a ball?



another representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$ with A square and nonsingular

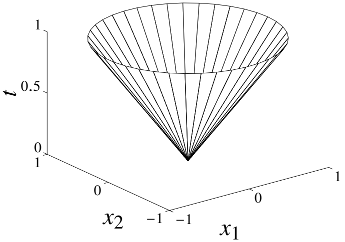
Norm balls and norm cones

- **norm:** a function $\|\cdot\|$ that satisfies
  - $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
  - $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
  - $\|x + y\| \leq \|x\| + \|y\|$
- notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm
- **norm ball** with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$
- **norm cone:** $\{(x, t) \mid \|x\| \leq t\}$
- norm balls and cones are convex   why?

Euclidean norm cone

$$\{(x, t) \mid \|x\|_2 \leq t\} \subset \mathbf{R}^{n+1}$$

is called **second-order cone**



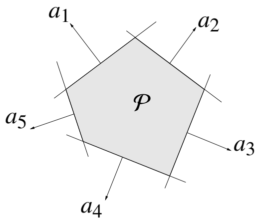
Polyhedra

- **polyhedron** is solution set of finitely many linear inequalities and equalities

$$\{x \mid Ax \preceq b,\ Cx = d\}$$

  ($A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)
- intersection of finite number of halfspaces and hyperplanes
- example with no equality constraints; a_i^T are rows of A

could be unbounded.



Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite (symmetric) $n \times n$ matrices

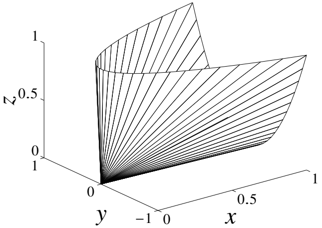
$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

- $\mathbf{S}^n_+$ is a convex cone, the **positive semidefinite cone**

why convex? show from definitions.

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite (symmetric) $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$



x+z >=0;
xz-y^2 >=0.

Outline

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Operations that preserve convexity

Generalized inequalities

Separating and supporting hyperplanes

Showing a set is convex

methods for establishing convexity of a set C

1. apply definition: show $x_1, x_2 \in C,\ 0 \le \theta \le 1 \implies \theta x_1 + (1-\theta)x_2 \in C$
   - recommended only for **very simple** sets

2. use convex functions (next lecture)

3. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity
   - intersection
   - affine mapping
   - perspective mapping
   - linear-fractional mapping   often seen in control..

you'll mostly use methods 2 and 3

Intersection

- the intersection of (any number of) convex sets is convex
- **example:**

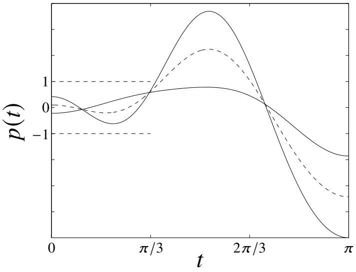
  p(x,t)

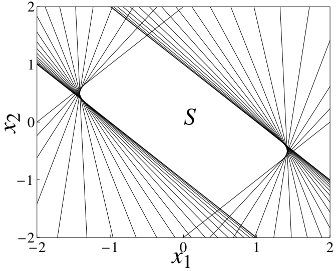
  – $S = \{x \in \mathbf{R}^m \mid |p(t)| \le 1 \text{ for } |t| \le \pi/3\}$, with $p(t) = x_1 \cos t + \cdots + x_m \cos mt$

  – write $S = \bigcap_{|t| \le \pi/3} \{x \mid |p(t)| \le 1\}$, *i.e.*, an intersection of (convex) slabs

  intersection of an uncountable number of convex sets.

- picture for $m = 2$:





each line is
p(x,t)=+1 or -1
for a fixed t,
of the form
p(x,t) = a1 x1 + a2 x2

Affine mappings

- suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine, *i.e.*, $f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$
- the **image** of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

why? make sure you check this.

- the **inverse image** $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

inverse image can increase dimension.

Examples

b is a vector.

- scaling, translation: $aS + b = \{ax + b \mid x \in S\}$, $a, b \in \mathbf{R}$
- projection onto some coordinates: $\{x \mid (x, y) \in S\}$
- if $S \subseteq \mathbf{R}^n$ is convex and $c \in \mathbf{R}^n$, $c^T S = \{c^T x \mid x \in S\}$ is an interval
- solution set of **linear matrix inequality** $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ with $A_i, B \in \mathbf{S}^p$
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\}$ with $P \in \mathbf{S}^n_+$

why? book: inverse image of positive semi-definite cone under f(x)=B-A(x)

why? book: inverse image of second-order cone z^T z <= t^2 under affine function f(x) = (P^1/2 x, c^T x)

Perspective and linear-fractional function

- **perspective function** $P : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

image of a pin-hole camera with hole at origin, white screen at x2 =-1.

- images and inverse images of convex sets under perspective are convex

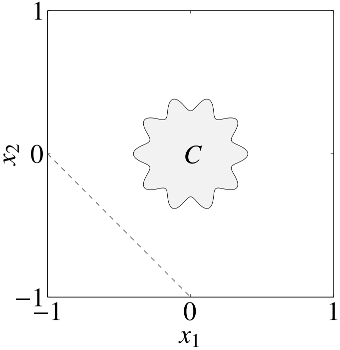
- **linear-fractional function** $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$:

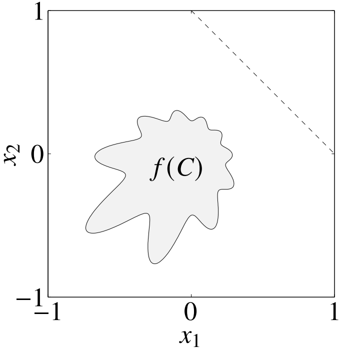
$$f(x) = \frac{Ax+b}{c^Tx+d}, \qquad \mathbf{dom}\, f = \{x \mid c^Tx + d > 0\}$$

- images and inverse images of convex sets under linear-fractional functions are convex

Linear-fractional function example

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$





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Proper cones

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

how would you show this?
Intersection of linear inequalities.

Generalized inequality

- (nonstrict and strict) **generalized inequality** defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \mathbf{int}\, K$$

- **examples** a partial ordering.
  - componentwise inequality ($K = \mathbf{R}^n_+$): $x \preceq_{\mathbf{R}^n_+} y \iff x_i \leq y_i, \quad i = 1, \ldots, n$
  - matrix inequality ($K = \mathbf{S}^n_+$): $X \preceq_{\mathbf{S}^n_+} Y \iff Y - X$ positive semidefinite

  these two types are so common that we drop the subscript in $\preceq_K$

- many properties of $\preceq_K$ are similar to $\leq$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$$

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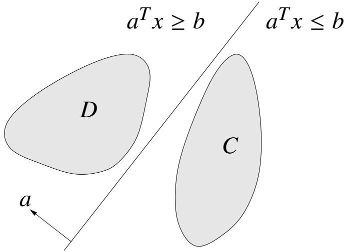
Separating hyperplane theorem

VERY important property of convex sets!

- if C and D are nonempty disjoint (*i.e.*, $C \cap D = \emptyset$) convex sets, there exist $a \neq 0$, b s.t.

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



- the hyperplane $\{x \mid a^T x = b\}$ **separates** C and D

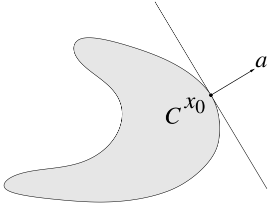
example where there is no strict separation?

- strict separation requires additional assumptions (*e.g.*, C is closed, D is a singleton)

Supporting hyperplane theorem

- suppose x_0 is a boundary point of set $C \subset \mathbf{R}^n$
- **supporting hyperplane** to C at x_0 has form $\{x \mid a^T x = a^T x_0\}$, where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

along the direction of a.



- **supporting hyperplane theorem:** if C is convex, then there exists a supporting hyperplane at every boundary point of C